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Surface waves in hot plasmas

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Abstract. Vlasov's equation and the full set of Maxwell's equations are solved as an initial value problem in a semi-infinite plasma. On specifying boundary conditions, a dispersion relation is obtained for surface waves in two situations, one without a wave incident on the boundary, the other with such a wave. The former case includes all previous results as special cases. In the latter case, we find that surface waves cannot be excited by a wave incident on the boundary.

1. Introduction

Plasma boundaries are usually regarded as sources of mathematical complication in studies of wave propagation in plasmas. Landau (1946), in his paper on the propagation and damping of electrostatic plasma oscillations, also examined the penetration of an electromagnetic wave into a plasma half-space. There have been many subsequent studies of this class of problems, and Clemmow and Karunarathne (1970) have referred to much of this work. There is however a different situation in which boundaries, far from being merely a complication in the analysis, are essential to wave propagation. It has been known for some time that plasmas with vacuum or dielectric boundaries support waves which propagate along the surface of the plasma in addition to those which propagate within the plasma (Allis *et al* 1963).

Recently interest in surface plasma waves has revived, partly on account of widespread laboratory studies of the interaction between high power lasers and plasmas. Early work by Trivelpiece and Gould (1959) was restricted to an examination of electrostatic surface waves on cold homogeneous plasmas which showed characteristic oscillations at the frequency $\omega_p/\sqrt{2}$. Ritchie (1963) was the first to allow for temperature effects on the surface wave dispersion relation in a study of surface oscillations in metal foils. He applied the hydrodynamic electron theory of Bloch to an electron gas of uniform density and obtained a thermal correction to the cold plasma dispersion relation that was quite different from the corresponding contribution to the dispersion relation for bulk plasma waves.

As in the case of bulk plasma waves one would expect a kinetic theory study based on the Vlasov equation to provide information not only about the propagation of plasma waves but also to determine the collisionless damping, if any. This has been carried out by Guernsey (1969), for the electrostatic limit, in work on a semi-infinite homogeneous plasma confined by a perfectly reflecting wall. An initial value problem was solved in the half-space using the (nonrelativistic) Vlasov equation. Guernsey found that the least damped contributions to the electric field in the plasma are of two types. One corresponds to the bulk plasma waves obeying the familiar Landau dispersion relation.

The second contribution describes surface waves and Guernsey has given a dispersion relation for these and evaluated it in the long wavelength limit, where he found that both the dispersion and the rate of damping of surface waves greatly exceed the corresponding quantities for bulk plasma waves.

In all the work so far described, dispersion relations have been found only in the electrostatic limit. This restriction was relaxed in work by Vedenov (1965) who gave a dispersion relation for surface waves on a cold, homogeneous (metal) plasma with a plane boundary, using the full set of Maxwell's equations. Recently Kaw and McBride (1970) have found an equivalent relation for a temperate plasma using the fluid equations for a homogeneous plasma confined in a half-space.

The present calculation adopts a kinetic theory approach, and treats the most general case by solving, as an initial value problem, Vlasov's equation with the full set of Maxwell's equations. This is done in § 2. Section 3 demonstrates how, on specifying the boundary conditions, the surface wave dispersion relation arises as a natural consequence of solving the initial value problem. We have obtained explicit dispersion relations for nonradiating surface waves and for the case of a wave incident on the boundary. Section 4 evaluates the dispersion relation for the former case for cold and warm plasmas and compares it with previous results.

2. The solution of Vlasov's equation in a plasma half-space

We consider a multicomponent plasma which occupies the region $z \geqslant 0$. Vlasov's equation for a species s, with charge e_s and mass m_s, is

$$\frac{\partial f^s}{\partial t}+\boldsymbol{v}\cdot\frac{\partial f^s}{\partial \boldsymbol{r}}+\frac{e_s}{m_s}\boldsymbol{E}(\boldsymbol{r},t)\cdot\frac{\partial f^s}{\partial \boldsymbol{v}} = 0.$$

Assuming a small perturbation about a homogeneous isotropic equilibrium distribution $f_0^s(v)$, that is

$$f^s(\boldsymbol{r},\boldsymbol{v},t) = f_0^s(v)+f_1^s(\boldsymbol{r},\boldsymbol{v},t)$$

where $|f_1^s| \ll f_0$, the Vlasov equation gives

$$\frac{\partial f_1^s}{\partial t}+\boldsymbol{v}\cdot\frac{\partial f_1^s}{\partial \boldsymbol{r}} = -\frac{e_s}{m_s}\boldsymbol{E}(\boldsymbol{r},t)\cdot\frac{\partial f_0^s}{\partial \boldsymbol{v}}. \tag{1}$$

From Maxwell's equations one has, as usual

$$\nabla\times(\nabla\times\boldsymbol{E})+\frac{1}{c^2}\frac{\partial^2\boldsymbol{E}}{\partial t^2}+\frac{4\pi}{c^2}\frac{\partial \boldsymbol{j}}{\partial t} = 0 \tag{2}$$

where

$$\boldsymbol{j} = \sum_s n_{0s}e_s\int \mathrm{d}\boldsymbol{v}\,\boldsymbol{v}f_1^s(\boldsymbol{r},\boldsymbol{v},t).$$

We must specify a boundary condition for the distribution function at $z = 0$. In a physical situation the zero order distribution function would be inhomogeneous increasing from zero at the boundary. Unfortunately such a situation does not lend itself to an analytic treatment and we are therefore forced to consider an idealized boundary. Two models are commonly used: either a diffuse boundary such that $f_1^s = 0$

on $z = 0$ in which the particles, arriving at the boundary, are scattered with complete loss of their drift velocity; or, more usually, a boundary where each species is reflected specularly, that is, $f_1^s(v_z, z = 0) = f_1^s(-v_z, z = 0)$. Reuter and Sondheimer (1949) in their treatment of skin effects in metals used a combination of the two boundary conditions. Since it lends itself to the easier analytic treatment, we will use the specular reflection condition.

This can be achieved by defining f_1^s for $z < 0$ such that

$$f_1^s(-z, v_z) = f_1^s(z, -v_z).$$

Vlasov's equation, (1), can then be made invariant in the transformation ($z \to -z$, $v_z \to -v_z$) if $\boldsymbol{E}$ is defined for $z < 0$ such that $\boldsymbol{E}_{\parallel}$ (parallel to the surface $z = 0$) is even in z and E_z is odd. Then equation (1) is satisfied for all z.

Taking Fourier transforms in space and Laplace transforms in time, that is, applying

$$\int_{-\infty}^{\infty} \mathrm{d}\boldsymbol{r} \exp(-\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r}) \int_0^{\infty} \mathrm{d}t \exp(\mathrm{i}\omega t)$$

to equations (1) and (2), one has

$$f_1^s(\boldsymbol{k}, \boldsymbol{v}, \omega) = \left(-\frac{e_s}{m_s} \boldsymbol{E}(\boldsymbol{k}, \omega) \cdot \frac{\partial f_0^s}{\partial \boldsymbol{v}} + f_1^s(\boldsymbol{k}, \boldsymbol{v}, 0) \right) \frac{1}{\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{v} - \omega)} \tag{3}$$

and

$$\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{E}) + \frac{\omega^2}{c^2} \boldsymbol{E} + \frac{4\pi \mathrm{i}\omega}{c^2} \boldsymbol{j} = \boldsymbol{M}_{\parallel} + \boldsymbol{R} \tag{4}$$

where

$$\boldsymbol{M} = 2\frac{\partial \boldsymbol{E}(0, \boldsymbol{k}_{\parallel}, \omega)}{\partial z} - 2\mathrm{i}E_z(0, \boldsymbol{k}_{\parallel}, \omega)\boldsymbol{k}$$

$$\boldsymbol{R} = \frac{1}{c^2}\left(\mathrm{i}\omega \boldsymbol{E}(\boldsymbol{k}, 0) - \frac{\partial \boldsymbol{E}(\boldsymbol{k}, 0)}{\partial t} - 4\pi \boldsymbol{j}(\boldsymbol{k}, 0) \right).$$

Now

$$4\pi \boldsymbol{j}(\boldsymbol{k}, \omega) = 4\pi \sum_s n_{0s} e_s \int \mathrm{d}\boldsymbol{v}\, \boldsymbol{v} f_1^s(\boldsymbol{k}, \boldsymbol{v}, \omega)$$

$$= \mathrm{i} \sum_s \omega_{\mathrm{ps}}^2 \boldsymbol{E}(\boldsymbol{k}, \omega) \cdot \int \frac{\mathrm{d}\boldsymbol{v}\, \boldsymbol{v}(\partial f_0^s / \partial \boldsymbol{v})}{\boldsymbol{k} \cdot \boldsymbol{v} - \omega} - 4\pi \mathrm{i} \sum_s n_{0s} e_s \int \frac{\mathrm{d}\boldsymbol{v}\, \boldsymbol{v} f_1^s(\boldsymbol{k}, \boldsymbol{v}, 0)}{\boldsymbol{k} \cdot \boldsymbol{v} - \omega}$$

where

$$\omega_{\mathrm{ps}}^2 = \frac{4\pi n_{0s} e_s^2}{m_s}.$$

Separating the field into its longitudinal and transverse parts, that is $\boldsymbol{E} = \boldsymbol{E}^{\mathrm{T}} + E^{\mathrm{L}}(\boldsymbol{k}/k)$, we get

$$-\left(k^2 - \frac{\omega^2}{c^2}\right)\boldsymbol{E}^{\mathrm{T}} + \frac{\omega^2}{c^2} E^{\mathrm{L}} \frac{\boldsymbol{k}}{k} - \frac{\omega}{c^2} \sum_s \omega_{\mathrm{ps}}^2 \left(\boldsymbol{E}^{\mathrm{T}} + E^{\mathrm{L}} \frac{\boldsymbol{k}}{k} \right) \cdot \int \frac{\mathrm{d}\boldsymbol{v}\, \boldsymbol{v}(\partial f_0^s / \partial \boldsymbol{v})}{\boldsymbol{k} \cdot \boldsymbol{v} - \omega}$$

$$= \boldsymbol{M}_{\parallel} + \boldsymbol{R} - \frac{4\pi \mathrm{i}\omega}{c^2} \sum_s n_{0s} e_s \int \frac{\mathrm{d}\boldsymbol{v}\, \boldsymbol{v} f_1^s(\boldsymbol{k}, \boldsymbol{v}, 0)}{\boldsymbol{k} \cdot \boldsymbol{v} - \omega}.$$

But

$$\int \frac{\mathrm{d}\boldsymbol{v}\, \boldsymbol{v}\boldsymbol{E}^{\mathrm{T}} . (\partial f_0^s/\partial \boldsymbol{v})}{\boldsymbol{k} . \boldsymbol{v} - \omega} = -\boldsymbol{E}^{\mathrm{T}} \int \frac{\mathrm{d}\boldsymbol{v} f_0^s}{\boldsymbol{k} . \boldsymbol{v} - \omega}.$$

Thus

$$\frac{\omega^2}{c^2} \Delta^{\mathrm{L}} E^{\mathrm{L}} \frac{\boldsymbol{k}}{k} - \Delta^{\mathrm{T}} \boldsymbol{E}^{\mathrm{T}} = \boldsymbol{S} \tag{5}$$

where

$$\Delta^{\mathrm{L}}(\boldsymbol{k}, \omega) = 1 - \sum_s \omega_{\mathrm{ps}}^2 \frac{\boldsymbol{k}}{k^2} . \int \frac{\mathrm{d}\boldsymbol{v} (\partial f_0^s/\partial \boldsymbol{v})}{\boldsymbol{k} . \boldsymbol{v} - \omega} \tag{6}$$

$$\Delta^{\mathrm{T}}(\boldsymbol{k}, \omega) = k^2 - \frac{\omega^2}{c^2} - \frac{\omega}{c^2} \sum_s \omega_{\mathrm{ps}}^2 \int \frac{\mathrm{d}\boldsymbol{v} f_0^s}{\boldsymbol{k} . \boldsymbol{v} - \omega} \tag{7}$$

$$\boldsymbol{S} = \boldsymbol{M}_{\parallel} + \boldsymbol{R} - \frac{4\pi\omega}{c^2} \sum_s n_{0s} e_s \int \frac{\mathrm{d}\boldsymbol{v}\, \boldsymbol{v} f_1^s(\boldsymbol{k}, \boldsymbol{v}, 0)}{\boldsymbol{k} . \boldsymbol{v} - \omega}.$$

Hence

$$E^{\mathrm{L}}(\boldsymbol{k}, \omega) = \frac{1}{\Delta^{\mathrm{L}}} \left(\frac{c^2}{\omega^2} \right) \left(\frac{\boldsymbol{k} . \boldsymbol{S}}{k} \right) \tag{8}$$

$$\boldsymbol{E}^{\mathrm{T}}(\boldsymbol{k}, \omega) = \frac{1}{\Delta^{\mathrm{T}}} \frac{\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{S})}{k^2}. \tag{9}$$

Note that $\Delta^{\mathrm{L}} = 0$ is the usual electrostatic plasma wave dispersion relation and $\Delta^{\mathrm{T}} = 0$ that for transverse electromagnetic waves.

Isolating the initial conditions in (8) and (9), we have

$$E^{\mathrm{L}}(\boldsymbol{k}, \omega) = \frac{1}{\Delta^{\mathrm{L}}} \left(\frac{c^2}{\omega^2} \right) \frac{\boldsymbol{k}_{\parallel} . \boldsymbol{M}_{\parallel}}{k} + \frac{1}{\Delta^{\mathrm{L}}} \left(\frac{c^2}{\omega^2} \right) \left(\frac{\boldsymbol{k} . \boldsymbol{T}}{k} \right) \tag{10}$$

$$\boldsymbol{E}^{\mathrm{T}}(\boldsymbol{k}, \omega) = \frac{1}{\Delta^{\mathrm{T}}} \frac{\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{M}_{\parallel})}{k^2} + \frac{1}{\Delta^{\mathrm{T}}} \frac{\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{T})}{k^2} \tag{11}$$

where

$$\boldsymbol{T} = \boldsymbol{R} - \frac{4\pi\omega}{c^2} \sum_s n_{0s} e_s \int \frac{\mathrm{d}\boldsymbol{v}\, \boldsymbol{v} f_1^s(\boldsymbol{k}, \boldsymbol{v}, 0)}{\boldsymbol{k} . \boldsymbol{v} - \omega}$$

contains all the initial conditions and $\boldsymbol{M}$ all the boundary conditions on $z = 0$.

3. The surface wave dispersion relation

To determine the surface wave dispersion relation one must specify the boundary conditions on the field vectors at $z = 0$. These are taken to be the continuity of the electric and magnetic fields parallel to the surface. Consider the case where the surface mode does not radiate, that is, the electric field in free space is given by

$$\boldsymbol{E}^{\mathrm{v}} = \boldsymbol{C} \exp(k_{\perp} z) \qquad k_{\perp} = \left(k_{\parallel}^2 - \frac{\omega^2}{c^2} \right)^{1/2}$$

(note: the x, y, t dependence is assumed to be $\exp \mathrm{i}(\boldsymbol{k}_{\parallel} . \boldsymbol{r}_{\parallel} - \omega t)$).

Six equations are required to determine the unknowns $\boldsymbol{C}$, $\partial \boldsymbol{E}(0, \boldsymbol{k}_{\parallel}, \omega)/\partial z$, $E_z(0, \boldsymbol{k}_{\parallel}, \omega)$. In free space, Poisson's equation gives

$$\mathrm{i}\boldsymbol{k}_{\parallel} \cdot \boldsymbol{C}_{\parallel} + k_{\perp} C_z = 0 \tag{12}$$

while in the plasma

$$\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{E} = 4\pi \sum_s n_{0s} e_s \int f_1^s \, \mathrm{d}\boldsymbol{v} + 2E_z(0, \boldsymbol{k}_{\parallel}, \omega)$$

therefore

$$ik\Delta^{\mathrm{L}} E^{\mathrm{L}} - \mathrm{i} \sum_s \omega_{ps}^2 \boldsymbol{E}^{\mathrm{T}} \cdot \int \frac{(\partial f_0^s/\partial \boldsymbol{v})\,\mathrm{d}\boldsymbol{v}}{\boldsymbol{k} \cdot \boldsymbol{v} - \omega} = 2E_z(0, \boldsymbol{k}_{\parallel}, \omega) - 4\pi \mathrm{i} \sum_s n_{0s} e_s \int \frac{f_1^s(\boldsymbol{k}, \boldsymbol{v}, 0)\,\mathrm{d}\boldsymbol{v}}{\boldsymbol{k} \cdot \boldsymbol{v} - \omega}.$$

But

$$\boldsymbol{E}^{\mathrm{T}} \cdot \int \frac{(\partial f_0^s/\partial \boldsymbol{v})\,\mathrm{d}\boldsymbol{v}}{\boldsymbol{k} \cdot \boldsymbol{v} - \omega} \equiv 0$$

and so

$$E^{\mathrm{L}}(\boldsymbol{k}, \omega) = -\frac{\mathrm{i}}{k\Delta^{\mathrm{L}}} \left(2E_z(0, \boldsymbol{k}_{\parallel}, \omega) - 4\pi \mathrm{i} \sum_s n_{0s} e_s \int \frac{f_1^s(\boldsymbol{k}, \boldsymbol{v}, 0)\,\mathrm{d}\boldsymbol{v}}{\boldsymbol{k} \cdot \boldsymbol{v} - \omega} \right).$$

Equating this with (8), we get

$$\mathrm{i}\left(2E_z(0, \boldsymbol{k}_{\parallel}, \omega) - 4\pi \mathrm{i} \sum_s n_{0s} e_s \int \frac{f_1^s(\boldsymbol{k}, \boldsymbol{v}, 0)\,\mathrm{d}\boldsymbol{v}}{\boldsymbol{k} \cdot \boldsymbol{v} - \omega} \right) = -\frac{c^2}{\omega^2} \boldsymbol{k} \cdot \boldsymbol{S}. \tag{13}$$

The continuity of $\boldsymbol{E}_{\parallel}$ gives

$$\boldsymbol{C}_{\parallel} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k_z \left\{ \left(\frac{\boldsymbol{k}_{\parallel} \cdot \boldsymbol{M}_{\parallel}}{k} \right) \left(\frac{1}{\Delta^{\mathrm{T}}} + \frac{c^2}{\omega^2 \Delta^{\mathrm{L}}} \right) \frac{\boldsymbol{k}_{\parallel}}{k} - \frac{\boldsymbol{M}_{\parallel}}{\Delta^{\mathrm{T}}} \right\} + \boldsymbol{K}_{\parallel}. \tag{14}$$

The continuity of $\boldsymbol{B}_{\parallel}$ gives

$$\begin{aligned} k_{\perp} \boldsymbol{C}_{\parallel} - \mathrm{i} C_z \boldsymbol{k}_{\parallel} &= \frac{\mathrm{i}}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k_z \frac{1}{\Delta^{\mathrm{T}}} (S_z \boldsymbol{k}_{\parallel} - \boldsymbol{S}_{\parallel} k_z) \\ &= -\frac{\mathrm{i}}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}k_z k_z}{\Delta^{\mathrm{T}}} \boldsymbol{M}_{\parallel} + \boldsymbol{L}_{\parallel} \end{aligned} \tag{15}$$

where

$$\boldsymbol{K} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}k_z \left\{ \frac{1}{\Delta^{\mathrm{L}}} \left(\frac{c^2}{\omega^2} \right) \left(\frac{\boldsymbol{k} \cdot \boldsymbol{T}}{k} \right) \frac{\boldsymbol{k}}{k} - \frac{1}{\Delta^{\mathrm{T}}} \frac{\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{T})}{k^2} \right\}$$

$$\boldsymbol{L} = \frac{\mathrm{i}}{2\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}k_z}{\Delta^{\mathrm{T}}} (T_z \boldsymbol{k} - k_z \boldsymbol{T})$$

(note: where we write $\int_{-\infty}^{\infty} \mathrm{d}k_z$, it is to be understood that $\lim_{z \to 0+} \int_{-\infty}^{\infty} \mathrm{d}k_z \exp(\mathrm{i}k_z z)$ is intended).

The six required equations are (12)–(15). Equations (12), (14), (15) give respectively

$$\mathrm{i}\boldsymbol{k}_{\parallel}\,.\,\boldsymbol{C}_{\parallel}+k_{\perp}C_z=0$$

$$\boldsymbol{k}_{\parallel}\,.\,\boldsymbol{C}_{\parallel}+\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}k_z\left(\frac{k_z^2}{k^2\Delta^{\mathrm{T}}}-\frac{c^2k_{\parallel}^2}{\omega^2k^2\Delta^{\mathrm{L}}}\right)\right\}\boldsymbol{k}_{\parallel}\,.\,\boldsymbol{M}_{\parallel}=\boldsymbol{k}_{\parallel}\,.\,\boldsymbol{K}_{\parallel}$$

$$\mathrm{i}k_{\parallel}C_z-\frac{k_{\perp}}{k_{\parallel}}\boldsymbol{k}_{\parallel}\,.\,\boldsymbol{C}_{\parallel}-\left(\frac{\mathrm{i}}{2\pi k_{\parallel}}\int_{-\infty}^{\infty}\frac{\mathrm{d}k_zk_z}{\Delta^{\mathrm{T}}}\right)\boldsymbol{k}_{\parallel}\,.\,\boldsymbol{M}_{\parallel}=-\frac{\boldsymbol{k}_{\parallel}\,.\,\boldsymbol{L}_{\parallel}}{k_{\parallel}}.$$

The surface wave dispersion relation is then

$$\epsilon(k_{\parallel},\omega)=\begin{vmatrix}\mathrm{i} & k_{\parallel} & 0\\ 1 & 0 & \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}\mathrm{d}k_z\left(\frac{k_z^2}{k^2\Delta^{\mathrm{T}}}-\frac{c^2k_{\parallel}^2}{\omega^2k^2\Delta^{\mathrm{L}}}\right)\\ -\dfrac{k_{\perp}}{k_{\parallel}} & \mathrm{i}k_{\parallel} & -\dfrac{\mathrm{i}}{2\pi k_{\parallel}}\displaystyle\int_{-\infty}^{\infty}\frac{\mathrm{d}k_zk_z}{\Delta^{\mathrm{T}}}\end{vmatrix}=0.$$

Thus

$$\epsilon(k_{\parallel},\omega)=\frac{1}{2\pi k_{\parallel}}\int_{-\infty}^{\infty}\mathrm{d}k_z\left(\frac{k_{\parallel}^2}{k^2\Delta^{\mathrm{L}}}-\frac{\omega^2k_z^2}{c^2k^2\Delta^{\mathrm{T}}}-\frac{\mathrm{i}k_{\perp}k_z}{\Delta^{\mathrm{T}}}\right) \tag{16}$$

or

$$\epsilon(k_{\parallel},\omega)=\frac{k}{2\pi k_{\parallel}}\int_{-\infty}^{\infty}\frac{\mathrm{d}k_z(k_{\perp}-\mathrm{i}k_z)}{\Delta^{\mathrm{T}}}-\frac{k_{\parallel}}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}k_z}{k^2}\left(\frac{k^2-(\omega^2/c^2)}{\Delta^{\mathrm{T}}}-\frac{1}{\Delta^{\mathrm{L}}}\right). \tag{17}$$

In the electrostatic limit ($c\to\infty$), we have that $\Delta^{\mathrm{T}}\to k^2$

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}k_zk_z}{k^2}=\pi\mathrm{i}\qquad\int_{-\infty}^{\infty}\frac{\mathrm{d}k_z}{k^2}=\frac{\pi}{k_{\parallel}}$$

and so, from (17)

$$\epsilon(k_{\parallel},\omega)=1-\frac{k_{\parallel}}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}k_z}{k^2}\left(1-\frac{1}{\Delta^{\mathrm{L}}}\right) \tag{18}$$

This expression is identical to that derived by Guernsey (his equation (16)).

Let us now consider the case where we have a plane wave incident on the boundary, that is, $\boldsymbol{E}_{\mathrm{i}}^{\mathrm{v}}=\boldsymbol{C}\exp(\mathrm{i}k_z^{\mathrm{v}}z)$ where $k_z^{\mathrm{v}}=\{(\omega^2/c^2)-k_{\parallel}^2\}^{1/2}$ (or, inversely, that the surface wave radiates). Then, including the reflected wave, we have

$$\boldsymbol{E}^{\mathrm{v}}=\boldsymbol{C}\exp(\mathrm{i}k_z^{\mathrm{v}}z)+\boldsymbol{D}\exp(-\mathrm{i}k_z^{\mathrm{v}}z)$$

where $\boldsymbol{C}$ is prescribed and $\boldsymbol{D}$ unknown. The boundary conditions give, as before

$$\mathrm{i}\boldsymbol{k}_{\parallel}\,.\,(\boldsymbol{C}_{\parallel}+\boldsymbol{D}_{\parallel})+\mathrm{i}k_z^{\mathrm{v}}(C_z-D_z)=0 \tag{12a}$$

$$\boldsymbol{C}_{\parallel}+\boldsymbol{D}_{\parallel}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}k_z\left\{\left(\frac{\boldsymbol{k}_{\parallel}\,.\,\boldsymbol{M}_{\parallel}}{k}\right)\left(\frac{1}{\Delta^{\mathrm{T}}}+\frac{c^2}{\omega^2\Delta^{\mathrm{L}}}\right)\frac{\boldsymbol{k}_{\parallel}}{k}-\frac{\boldsymbol{M}_{\parallel}}{\Delta^{\mathrm{T}}}\right\}+\boldsymbol{K}_{\parallel} \tag{14a}$$

$$\mathrm{i}k_z^{\mathrm{v}}(\boldsymbol{C}_{\parallel}-\boldsymbol{D}_{\parallel})-\mathrm{i}(C_z+D_z)\boldsymbol{k}_{\parallel}=-\frac{\mathrm{i}}{2\pi}\boldsymbol{M}_{\parallel}\int_{-\infty}^{\infty}\frac{\mathrm{d}k_zk_z}{\Delta^{\mathrm{T}}}+\boldsymbol{L}_{\parallel}. \tag{15a}$$

Equation (13) does not change. Solving for $\boldsymbol{M}_{\parallel}$ by eliminating $\boldsymbol{C}_{\parallel}+\boldsymbol{D}_{\parallel}$ between (12*a*) and (14*a*), $\boldsymbol{D}_{\|}$ between (14*a*) and (15*a*), and D_z between the resulting two equations, we get

$$\boldsymbol{M}_{\|}\left(\frac{1}{2\pi}k_z^{\mathrm{v}}\int_{-\infty}^{\infty}\frac{\mathrm{d}k_z(k_z+k_z^{\mathrm{v}})}{\Delta^{\mathrm{T}}}\right)+\boldsymbol{k}_{\|}\,.\,\boldsymbol{M}_{\|}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}k_z}{k^2}\left(\frac{k^2-(\omega^2/c^2)}{\Delta^{\mathrm{T}}}-\frac{1}{\Delta^{\mathrm{L}}}\right)\boldsymbol{k}_{\|}\right\}=-\boldsymbol{P}_{\|} \tag{19}$$

where

$$\boldsymbol{P}_{\|}=2k_z^{\mathrm{v}}(k_z^{\mathrm{v}}\boldsymbol{C}_{\|}-C_z\boldsymbol{k}_{\|})-\mathrm{i}k_z^{\mathrm{v}}\boldsymbol{L}_{\|}+k_z^{\mathrm{v}2}\boldsymbol{K}_{\|}-(\boldsymbol{k}_{\|}\,.\,\boldsymbol{K}_{\|})\boldsymbol{k}_{\|}\,.$$

Then

$$\boldsymbol{k}_{\|}\,.\,\boldsymbol{M}_{\|}=\frac{-1}{\epsilon(k_{\|},\omega)}\left(\frac{\boldsymbol{k}_{\|}\,.\,\boldsymbol{P}_{\|}}{k_{\|}}\right) \tag{20}$$

$$\boldsymbol{M}_{\|}=\boldsymbol{k}_{\|}\times(\boldsymbol{k}_{\|}\times\boldsymbol{P}_{\|})\left(\frac{k_z^{\mathrm{v}}k_{\|}^2}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}k_z(k_z+k_z^{\mathrm{v}})}{\Delta^{\mathrm{T}}}\right)^{-1}-\frac{1}{\epsilon(k_{\|},\omega)}\left(\frac{\boldsymbol{k}_{\|}\,.\,\boldsymbol{P}_{\|}}{k_{\|}}\right)\frac{\boldsymbol{k}_{\|}}{k_{\|}^2} \tag{21}$$

where

$$\epsilon(k_{\|},\omega)=-\frac{k_z^{\mathrm{v}}}{2\pi k_{\|}}\int_{-\infty}^{\infty}\frac{\mathrm{d}k_z(k_z+k_z^{\mathrm{v}})}{\Delta^{\mathrm{T}}}-\frac{k_{\|}}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}k_z}{k^2}\left(\frac{k^2-(\omega^2/c^2)}{\Delta^{\mathrm{T}}}-\frac{1}{\Delta^{\mathrm{L}}}\right). \tag{22}$$

This is identical with equation (17), the only difference being notational: $\mathrm{i}k_\perp = k_z^{\mathrm{v}}$.

4. Approximate dispersion relation for warm plasmas

For simplicity, we will consider a maxwellian electron plasma. Then

$$\Delta^{\mathrm{L}}=1+\frac{\omega_{\mathrm{p}}^2}{\omega^2}2y^2(1+yZ(y))$$

$$\Delta^{\mathrm{T}}=k^2-\frac{\omega^2}{c^2}-\frac{\omega_{\mathrm{p}}^2}{c^2}yZ(y)$$

where $y=\omega/V_{\mathrm{e}}k$, $V_{\mathrm{e}}=(2\kappa T/m)^{1/2}$; $Z(y)=(1/\sqrt{\pi})\int_{\mathrm{c}}\mathrm{e}^{-z^2}\,\mathrm{d}z/(z-y)$ is the usual plasma dispersion function. The asymptotic expansions of $Z(y)$ are such that

$$yZ(y)\simeq -1-\frac{1}{2y^2}\left(1+\frac{3}{2y^2}\right) \qquad y\gg 1$$

$$\simeq -2y^2 \qquad y\ll 1.$$

So for $y\gg 1$

$$\Delta^{\mathrm{L}}\simeq 1-\frac{\omega_{\mathrm{p}}^2}{\omega^2}\left(1+\frac{3V_{\mathrm{e}}^2k^2}{2\omega^2}\right)$$

$$\Delta^{\mathrm{T}}\simeq k^2-\frac{\omega^2}{c^2}+\frac{\omega_{\mathrm{p}}^2}{c^2}\left(1+\frac{V_{\mathrm{e}}^2k^2}{2\omega^2}\right).$$

Substituting these asymptotes into (17), we get

$$\omega^2 k_\parallel^2 = \left(k_\parallel^2 + \frac{2}{3V_e^2}\frac{\omega^2}{\omega_p^2}(\omega_p^2-\omega^2)\right)^{1/2}\left\{\omega^2\left(k_\parallel^2+\frac{\omega_p^2-\omega^2}{c^2}\right)^{1/2} - (\omega_p^2-\omega^2)\left(k_\parallel^2-\frac{\omega^2}{c^2}\right)^{1/2}\right\} \tag{23}$$

where we have taken $\omega_p^2 V_e^2/\omega^2 c^2 \ll 1$. This result can be compared directly with that of Kaw and McBride.

The limitation of the above analysis is thåt it uses the long wavelength asymptotes ($y \gg 1$) for Δ^L and Δ^T over the whole range of integration of k_z. Of course as $k_z \to \infty$, this asymptote becomes invalid, but we still might expect the error to be small since the integrand becomes small in this region.

To obtain a more accurate solution, we can split the range of integration into two regions: $0 < y < 1$ and $1 < y < \infty$ and use both asymptotes. We assume, as does Guernsey, that the region $y \sim 1$ contributes negligibly to the integrals.

Note that $y = 1$ gives $k_z = \{(\omega^2/V_e^2) - k_\parallel^2\}^{1/2} = \gamma$ say. Taking $\omega_p^2 V_e^2/\omega^2 c^2 \ll 1$, evaluation of the integrals of (17) gives

(*a*)
$$2\int_\gamma^\infty \frac{dk_z}{k^2}\left(\frac{k^2-(\omega^2/c^2)}{\Delta^T} - \frac{1}{\Delta^L}\right) = \frac{\pi}{k_\parallel}\left\{1-\frac{2}{\pi}\tan^{-1}\left(\frac{\omega^2}{V_e^2 k_\parallel^2}-1\right)^{1/2}\right\} - \frac{\pi}{\{k_\parallel^2+(2\omega_p^2/V_e^2)\}^{1/2}}\left\{1-\frac{2}{\pi}\tan^{-1}\left(\frac{\omega^2-k_\parallel^2 V_e^2}{2\omega_p^2+k_\parallel^2 V_e^2}\right)^{1/2}\right\}$$

(*b*)
$$2\int_0^\gamma \frac{dk_z}{k^2}\left(\frac{k^2-(\omega^2/c^2)}{\Delta^T} - \frac{1}{\Delta^L}\right) = \frac{2}{\omega_p^2-\omega^2}\left\{\frac{\omega_p^2}{\{k_\parallel^2+(\omega_p^2-\omega^2)/c^2\}^{1/2}} \times \tan^{-1}\left(\frac{(\omega^2/V_e^2)-k_\parallel^2}{k_\parallel^2+(\omega_p^2-\omega^2)/c^2}\right)^{1/2} - \frac{\omega^2}{[k_\parallel^2+\{2\omega^2(\omega_p^2-\omega^2)/3\omega_p^2V_e^2\}]^{1/2}} \times \tan^{-1}\left(\frac{(\omega^2/V_e^2)-k_\parallel^2}{k_\parallel^2+\{2\omega^2(\omega_p^2-\omega^2)/3\omega_p^2V_e^2\}}\right)^{1/2}\right\}$$

(*c*)
$$2\int_\gamma^\infty \frac{dk_z}{\Delta^T} = \frac{1}{k_\parallel}\left\{\pi - 2\tan^{-1}\left(\frac{(\omega^2/V_e^2)-k_\parallel^2}{k_\parallel^2-(\omega^2/c^2)}\right)^{1/2}\right\}$$

(*d*)
$$2\int_0^\gamma \frac{dk_z}{\Delta^T} = \frac{2}{[k_\parallel^2+\{(\omega_p^2-\omega^2)/c^2\}]^{1/2}}\tan^{-1}\left(\frac{(\omega^2/V_e^2)-k_\parallel^2}{k_\parallel^2+\{(\omega_p^2-\omega^2)/c^2\}}\right)^{1/2}$$

(*e*)
$$\int_{-\infty}^{\infty}\frac{dk_z k_z}{\Delta^T} = \lim_{Z\to 0+}\int_{-\infty}^{\infty}\frac{dk_z k_z e^{ik_z Z}}{\Delta^T} = \pi i.$$

Combining the terms according to (17) we get, after some simple algebra

$$\begin{aligned}\epsilon(k_\parallel, \omega) = \frac{k_\perp}{k_\parallel}&\left\{1 - \frac{1}{\pi}\tan^{-1}\left(\frac{(\omega^2/V_e^2) - k_\parallel^2}{k_\parallel^2 - \omega^2/c^2}\right)^{1/2}\right\} \\ &-\frac{1}{2}\left\{1 - \frac{2}{\pi}\tan^{-1}\left(\frac{\omega^2}{V_e^2 k_\parallel^2} - 1\right)^{1/2}\right\} \\ &+\frac{[k_\parallel^2 + \{(\omega_p^2 - \omega^2)/c^2\}]^{1/2}}{k_\parallel\{1 - (\omega_p^2/\omega^2)\}}\frac{1}{\pi}\tan^{-1}\left(\frac{(\omega^2/V_e^2) - k_\parallel^2}{k_\parallel^2 + \{(\omega_p^2 - \omega^2)/c^2\}}\right)^{1/2} \\ &+\frac{1}{2\{1 + (2\omega_p^2/V_e^2 k_\parallel^2)\}^{1/2}}\left\{1 - \frac{2}{\pi}\tan^{-1}\left(\frac{\omega^2 - k_\parallel^2 V_e^2}{2\omega_p^2 + k_\parallel^2 V_e^2}\right)^{1/2}\right\} \\ &+\frac{\omega^2 k_\parallel}{(\omega_p^2 - \omega^2)[k_\parallel^2 + \{2\omega^2(\omega_p^2 - \omega^2)/3\omega_p^2 V_e^2\}]^{1/2}} \\ &\times\frac{1}{\pi}\tan^{-1}\left(\frac{(\omega^2/V_e^2) - k_\parallel^2}{k_\parallel^2 + \{2\omega^2(\omega_p^2 - \omega^2)/3\omega_p^2 V_e^2\}}\right)^{1/2}. \end{aligned} \tag{24}$$

Assuming long wavelengths, that is $\omega/V_e k_\parallel \gg 1$, we get

$$\begin{aligned}\epsilon(k_\parallel, \omega) = &\left[\left(1 - \frac{\omega_p^2}{\omega^2}\right)\left(k_\parallel^2 - \frac{\omega^2}{c^2}\right)^{1/2} + \left(k_\parallel^2 + \frac{\omega_p^2 - \omega^2}{c^2}\right)^{1/2}\right]\left\{2k_\parallel\left(1 - \frac{\omega_p^2}{\omega^2}\right)\right\}^{-1} \\ &+\frac{k_\parallel}{\pi}\left[\frac{3}{2}\frac{\omega\omega_p V_e}{(\omega_p^2 - \omega^2)^{3/2}}\tan^{-1}\left\{\frac{2}{3}\left(1 - \frac{\omega^2}{\omega_p^2}\right)\right\}^{-1/2} \right. \\ &\left. +\frac{V_e}{\sqrt{2}\omega_p}\left(\frac{\pi}{2} - \tan^{-1}\frac{\omega}{\sqrt{2}\omega_p}\right) + \frac{V_e\omega}{\omega^2 - \omega_p^2}\right]. \end{aligned} \tag{25}$$

When the electrostatic limit is taken this expression reduces exactly to Guernsey's equation (36).

Figure 1 shows a semilog plot of the roots of the dispersion relation $\epsilon(k_\parallel, \omega) = 0$ *with ϵ given by (24) for temperatures of 0·01, 1, 100 eV and, for comparison, a plot of* Kaw and McBride's result (the broken curve) for 1 eV. The results are consistent in the electromagnetic region (small $k_\parallel$) but diverge in the electrostatic regime. We also note a spurious root at $\omega = \omega_p$ in Kaw and McBride's result which does not in fact exist (this can be seen either from (24), or from (23) which reduces to an identity as $\omega \to \omega_p$).

We note firstly that all the above results give, in the cold plasma limit

$$\frac{k_\parallel^2 c^2}{\omega^2} = \frac{\omega_p^2 - \omega^2}{\omega_p^2 - 2\omega^2}.$$

This is in agreement with the result of Vedenov and Kaw and McBride and gives $\omega = \omega_p/\sqrt{2}$ in the electrostatic limit as expected.

The differences in the results so far published appear only when we consider warm plasmas. Compare Kaw and McBride's equation (6) with our (23). We notice two differences (i) ω^2 for ω_p^2 on the left hand side and (ii) an extra factor ω^2/ω_p^2 in the first parentheses of (23). In the electrostatic limit ($c \to \infty$) (23) reduces to

$$\omega = \frac{\omega_p}{\sqrt{2}}\left(1 + \frac{\sqrt{3}}{2}\frac{k_\parallel}{k_e}\right) \qquad \left(k_e = \frac{\sqrt{2}\omega_p}{V_e}\right) \tag{26}$$

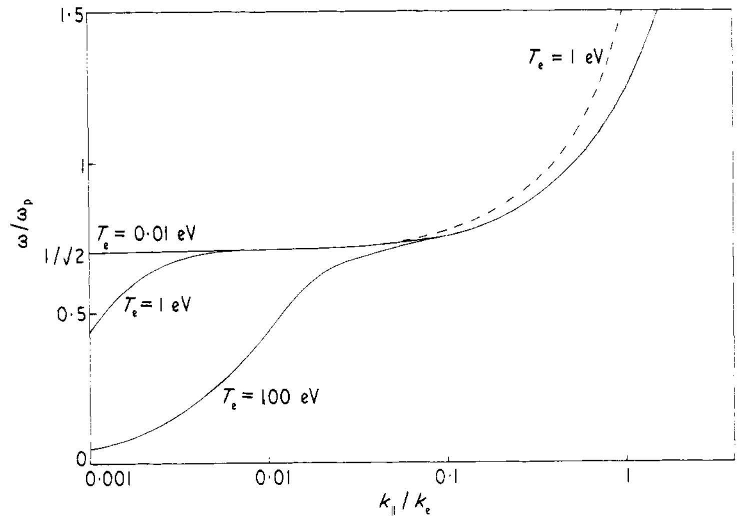

Figure 1. A semilog plot of the dispersion relation for surface plasma waves for temperatures of 0·01, 1, 100 eV. The broken curve is the Kaw and McBride result for 1 eV.

as compared with the

$$\omega = \frac{\omega_p}{\sqrt{2}}\left(1+\sqrt{\frac{3}{2}\frac{k_{\parallel}}{k_e}}\right) \tag{27}$$

of Kaw and McBride; (26) compares more favourably with the electrostatic limit of (25), that is, Guernsey's result

$$\omega = \frac{\omega_p}{\sqrt{2}}\left(1+0{\cdot}38\frac{k_{\parallel}}{k_e}\right).$$

Next, instead of specifying an initial value for the distribution function we investigate the response of a plasma to a wave incident on the boundary. In this case, ω and $k_{\parallel}$ are real. In the region $\omega < \omega_p$, the dispersion relation is evaluated as above and is seen to give no roots. When we consider the region $\omega > \omega_p$, the evaluation of the integrals is complicated due to there being roots of Δ^T and Δ^L on or near the contour of integration. Since the Laplace transform is strictly defined only for $\text{Im}\,\omega > 0$, it is necessary for situations where $\text{Im}\,\omega \leqslant 0$ to make an analytic continuation so that the dispersion relation is defined over the whole complex ω plane, and in our situation especially along the real ω line. This is achieved by deforming the contour such that we go around poles whenever $\text{Im}\,k_z > 0$. The result is that we again find no roots for the dispersion relation. This leads us to conclude that we cannot couple an external electromagnetic wave with a surface mode although it may do so with a bulk wave.

5. Discussion

We have obtained an initial value solution of the Vlasov equation and the full set of Maxwell's equations for a semi-infinite plasma assuming that the electrons are specularly reflected at the surface. This provides a general dispersion relation for surface waves which contains all the dispersion and damping characteristics of such waves and includes all previous work as special cases. In the cold plasma limit, we are in agreement with earlier results, namely that we find a mode which is a mixture of electromagnetic and space charge waves propagating in the frequency range 0 to $\omega_p/\sqrt{2}$; in the electrostatic limit, this degenerates into a surface oscillation of frequency $\omega_p/\sqrt{2}$. However, when we include finite temperature effects, certain disparities appear which lead to inaccurate dispersion characteristics, as was seen in the previous section. Comparison is most easily made in the electrostatic limit, where we see that values of the surface wave group velocity are overestimated.

The work presented here and that mentioned previously has been restricted to plasmas with sharp boundaries. There have been fewer studies of plasmas which are not spatially homogeneous, that is, those in which the density increases into the plasma from the boundary. Trivelpiece (1967) considered the effect of density gradients in the dispersion relation for electrostatic surface modes on a cylindrical plasma column. More recently, similar studies for a plasma half-space appear in the work of Kaw and McBride. They derive a full surface wave dispersion relation (ie not restricted to the electrostatic limit) for a cold plasma with a weak density profile and, in addition, that for electrostatic surface modes for a cold plasma with a linear density profile of arbitrary strength. However, if $\omega_p = 0$ at the surface, it was found that electrostatic surface waves do not propagate so that the presence of surface charge (and therefore the possibility of charge bunching) is essential for surface waves to exist.

The principal results of this work are:

(i) the general surface wave dispersion relation found ((17) and (22)) includes all earlier results. In the cold plasma limit it confirms a well known dispersion relation but highlights errors in warm plasma results appearing in the literature.

(ii) it does not appear to be possible to couple a radiation field into surface plasma waves.

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